# A note on the Menchov-Rademacher Inequality

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**Abstract**

We improve constants in the Rademacher-Menchov inequality by showing that

$$\mathbf{E}(\sup_{1\leqslant k\leqslant n} |\sum_{i=1}^{k} X_i|^2) \leqslant (a + b\log_2^2 n),$$

for all orthogonal random variables $X_1, ..., X_n$ such that $\sum_{k=1}^{n} \mathbf{E}|X_k|^2 = 1$.

## 1 Introduction

We consider real or complex orthogonal random variables $X_1, ..., X_n$, i.e.

$$\mathbf{E}|X_i|^2 < \infty, \ \ 1 \leqslant i \leqslant n \ \text{ and } \ \mathbf{E}(X_i X_j) = 0, \ \ 1 \leqslant i, j \leqslant n.$$

Let us denote $S_j := X_1 + ... + X_j$ for $1 \leqslant j \leqslant n$, and $S_0 = 0$. Clearly

$$\mathbf{E}|S_j - S_i|^2 = \sum_{k=i}^{j} \mathbf{E}|X_k|^2, \ \text{ for } i \leqslant j.$$

The best constant in the Menchov-Rademacher inequality is defined by

$$D_n := \sup \mathbf{E} \sup_{1\leqslant i\leqslant n} |S_i|^2,$$

where the supremum is taken over all orthogonal systems $X_1, ..., X_n$, which satisfy $\sum_{k=1}^{n} \mathbf{E}|X_k|^2 = 1$. We define also

$$C := \limsup_{n\to\infty} \frac{D_n}{\log_2^2 n}.$$

Rademacher [6] in 1922 and indepenedently Menchov [5] in 1923 proved that there exists $K > 0$ such that for $n \geqslant 2$

$$D_n \leqslant K \log_2^2 n, \ \text{ hence } \ C \leqslant K.$$

By now there are several different proofs of the above inequality. The traditional proof of Rademacher-Menchov inequality uses the bisection method (see Doob [1], and Loév [4]), which leads to

$$D_n \leqslant (2 + \log_2 n)^2, \ \ n \geqslant 2, \ \text{ hence } \ C \leqslant 1.$$

In 1970 Kounias [3] used a trisection method to get a finer inequality

$$D_n \leqslant (\frac{\log_2 n}{\log_2 3} + 2)^2, \ \ n \geqslant 2, \ \text{ hence } \ C \leqslant (\frac{\log_2 2}{\log_2 3})^2.$$

S. Chobayan, S.Levental and H. Salehi [2] proved the following result

$$D_{2n} \leqslant \frac{4}{3}D_n \ \text{ if } \ D_n \leqslant 3; \ \ D_{2n} \leqslant ((D_n - \frac{3}{4})^{1/2} + \frac{1}{2})^2 \tag{1}$$

and as a consequence they got the estimate $D_n \leqslant \frac{1}{4}(3 + \log_2^2 n)$, $C \leqslant \frac{1}{4}$. An example given in [2] shows that $D \geqslant \frac{\log_2^2 n}{\pi^2 \log_2^2 e}$ and thus $C \geqslant 0,04868$. The aim of this paper is to improve the bisection method and together with (1) to obtain that $C < \frac{1}{9}$.

## 2 Results

**Theorem 1** *For each $n, m \in \mathbb{N}$ and $l > 2$ the following inequality holds*

$$\sqrt{D_{n(2m+l)}} \leqslant \sqrt{D_n} + \sqrt{\max\{D_m, 2D_{l-1}\}}.$$

*If $l = 2$ then even stronger inequality holds true*

$$\sqrt{D_{n(2m+l)}} \leqslant \sqrt{D_n} + \sqrt{D_m}.$$

**Proof.** Let us denote $p := 2m + l$. The triangle inequality yields

$$|S_i| \leqslant |S_i - S_{pj}| + |S_{pj}|.$$

Consequently

$$\max_{1 \leqslant i \leqslant pn} |S_i| \leqslant \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{pj}| + \max_{0 \leqslant j \leqslant n} |S_{pj}|.$$

Thus

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} |S_i|^2 \leqslant \mathbf{E}(\max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{pj}| + \max_{0 \leqslant j \leqslant n} |S_{pj}|)^2.$$

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The definition of $D_n$ together with the classical norm inequality implies

$$\sqrt{D_{pn}} \leqslant \sqrt{D_n} + \sqrt{\mathbf{E} \max_i \min_{0 \leqslant j \leqslant n} |S_i - S_{pj}|^2}$$

It remains to show that

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{pj}|^2 \leqslant \max\{D_m, 2D_{l-1}\}, \quad \text{if } l > 2$$

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{pj}|^2 \leqslant D_m \quad \text{if } l = 2.$$

Let us denote

$$A_j := \max\{|S_i - S_{pj}| : \ pj \leqslant i \leqslant pj + m\},$$
$$B_j := \max\{|S_{p(j+1)} - S_i| : \ pj + m + l \leqslant i \leqslant p(j+1)\}$$
$$C_j := \max\{|S_i - S_{pj+m}| : \ pj + m < i < pj + m + l\}$$
$$D_j := \max\{|S_{pj+m+l} - S_i| : \ pj + m < i < pj + m + l\}$$

for each $j \in \{0, ..., n-1\}$. Each $0 \leqslant i \leqslant dn$ can be written in the form $i = pj + r$, where $j \in \{0, ..., n-1\}$, $r \in \{1, 2, ..., p\}$. If $r \leqslant m$, then

$$|S_i - S_{pj}|^2 \leqslant A_j^2.$$

If $r \geqslant m + l$

$$|S_{p(j+1)} - S_i|^2 \leqslant B_j^2.$$

The last case is when $i = pj + m + r$, $r \in \{1, ..., l-1\}$. Let us denote

$$P_j := S_{pj+m} - S_{pj}, \quad V_j := S_{pj+m+r} - S_{pj+m},$$
$$Q_j := S_{p(j+1)} - S_{pj+m+l}, \quad W_j := S_{pj+m+l} - S_{pj+m+r}.$$

Clearly $(i = pj + m + r, r \in \{1, ..., l-1\})$

$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} = \min\{|P_j + V_j|^2, |Q_j + W_j|^2\}.$$

For all complex numbers $a, b, c, d$ there is

$$\frac{1}{2}|a + b|^2 \leqslant |a|^2 + |b|^2, \quad \frac{1}{2}|c + d|^2 \leqslant |c|^2 + |d|^2.$$

Since

$$\min\{|a + b|^2, |c + d|^2\} \leqslant \frac{1}{2}|a + b|^2 + \frac{1}{2}|c + d|^2$$

we obtain that

$$\min\{|a + b|^2, |c + d|^2\} \leqslant |a|^2 + |b|^2 + |c|^2 + |d|^2.$$

Hence
$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} \leqslant |P_j|^2 + |Q_j|^2 + |V_j|^2 + |W_j|^2.$$

and consequently for each $pj < i \leqslant p(j+1)$ the following inequality holds

$$\min\{|S_i - S_{pj}|^2, |S_{p(j+1)} - S_i|^2\} \leqslant A_j^2 + B_j^2 + C_j^2 + D_j^2.$$

In fact we have proved that

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{2(m+1)j}|^2 \leqslant \mathbf{E} \sum_{j=0}^{n-1} (A_j^2 + B_j^2 + C_j^2 + D_j^2).$$

Let us observe that

$$\mathbf{E} A_j^2 \leqslant D_m \sum_{k=1}^{m} \mathbf{E}|X_{pj+k}|^2, \quad \mathbf{E} B_j^2 \leqslant D_m \sum_{k=1}^{m} \mathbf{E}|X_{pj+m+l+k}|^2,$$

$$\mathbf{E}(C_j^2 + D_j^2) \leqslant D_{l-1}(\mathbf{E}|X_{pj+m+1}|^2 + \mathbf{E}|X_{pj+m+l}|^2 + 2\sum_{k=2}^{l-1} \mathbf{E}|X_{pj+m+k}|^2) \ ,$$

Notice that if $l = 2$ then

$$\mathbf{E}(C_j^2 + D_j^2) \leqslant D_1(\mathbf{E}|X_{pj+m+1}|^2 + \mathbf{E}|X_{pj+m+1}|^2)$$

Hence, if $l > 2$

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{2(m+1)j}|^2 \leqslant \max\{D_m, 2D_{l-1}\}$$

and if $l = 2$

$$\mathbf{E} \max_{1 \leqslant i \leqslant pn} \min_{0 \leqslant j \leqslant n} |S_i - S_{2(m+1)j}|^2 \leqslant D_m.$$

It ends the proof.

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**Corollary 1** *For each $n \geqslant m$ the following inequality holds*

$$D_n \leqslant D_m(2 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)})^2.$$

**Proof.** Taking $l = 2$ in Theorem 1 we obtain

$$D_{m(2m+2)^k} \leqslant D_m(k+1)^2.$$

For each $n \geqslant m$ there exists $k \geqslant 0$ such that $m(2m+2)^{k-1} < n \leqslant m(2m+2)^k$. Hence

$$k < 1 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)}.$$

Consequently

$$D_n \leqslant D_m(2 + \frac{\log_2 n - \log_2 m}{\log_2(2m+2)})^2,$$

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The result implies

$$C = \limsup_{n \to \infty} \frac{D_n}{\log_2^2 n} \leqslant \frac{D_m}{\log_2^2(2m+2)}.$$

Putting $l > 2$ in Theorem 1 and proceeding we prove in the same way as in Corollary 1) we get the following result.

**Corollary 2** *For each $l > 2$ and $n \geqslant m$ the inequality holds true*

$$C \leqslant \frac{\max\{D_m, 2D_{l-1}\}}{\log_2^2(2m+l)}.$$

Let us remind that $D_2 = 4/3$. Hence applying Corollary 1 with $m = 2$ we get

$$C \leqslant \frac{4}{3\log_2^2 6} < \frac{1}{5}.$$

Observe that due to (1)

$$D_2 = \frac{4}{3}, \quad D_4 \leqslant (\frac{4}{3})^2, \quad D_8 \leqslant (\frac{4}{3})^3, \quad D_{16} \leqslant (\frac{4}{3})^4$$

and

$$D_{32} \leqslant ((\frac{4}{3})^4 - \frac{3}{4})^{1/2} + \frac{1}{2})^2, \quad D_{64} \leqslant (((D_{32} - \frac{3}{4})^{1/2} + \frac{1}{2})^2.$$

Hence

$$D_8 \leqslant 2,3704 \quad D_{64} \leqslant 5,5741.$$

Applying Corollary 2 with $m = 64$, $c = 9$ we obtain

$$C \leqslant 0,1107 < 1/9.$$

# References

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